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Optimal control of multiphase induction machine with third harmonic injection

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Abstract

This research report deals with deriving the optimal first+third harmonic current setpoints for different induction machine (IM) operation points. Since the rotor frequency and slip define the torque, the third harmonic virtual machine cannot be controlled optimally in the meaning of rotor flux-oriented control definition. Therefore the general torque equation is derived firstly for only the first harmonic control and then it is extended to the first+third harmonic control of multiphase IM.

Keywords

Multiphase machine, induction machine, optimal control

Contents

1 Introduction

The optimal control of multiphase machines with the injection of third harmonic currents is provided by the optimization introduced in [1]. In this report, the optimal control of multiphase ($m > 3$) induction machines (IM) is presented. In the beginning, the theory of first harmonic control is summarized and then the first and third harmonic control is discussed.

2 First Harmonic control

The stator and rotor flux linkages are defined as

$$\begin{aligned} \boldsymbol{\Psi_s} &= L_s \boldsymbol{i_s} + L_\mu \boldsymbol{i_r} \\ \boldsymbol{\Psi_r} &= L_\mu \boldsymbol{i_s} + L_r \boldsymbol{i_r} \end{aligned} \tag{1}$$

where $\boldsymbol{\Psi_s} = [\Psi_{ds}, \Psi_{qs}]^\top, \boldsymbol{i_s} = [i_{ds}, i_{qs}]^\top, \boldsymbol{i_r} = [i_{dr}, i_{qr}]^\top$. The stator and rotor inductances are defined as

$$\begin{aligned} L_s &= L_\mu + L_{s\sigma} \\ L_r &= L_\mu + L_{r\sigma} \end{aligned} \tag{2}$$

The torque produced by the induction machine is defined as

$$T = \frac{m p_p}{2} \Re \left\{ j \boldsymbol{\Psi}_s \boldsymbol{i}_s^* \right\} \tag{3}$$

which by decomposition to d-q reference frame equals

$$T = \frac{m p_p}{2} \Re \left\{ j \left(\Psi_{ds} + j \Psi_{qs} \right) \left(i_{ds} - j i_{qs} \right) \right\} = \frac{m p_p}{2} \left(\Psi_{ds} i_{qs} - \Psi_{qs} i_{ds} \right) \tag{4}$$

The flux linkages (1) are decomposed to d-q reference frame and substituted to (4)

$$T = \frac{m p_p}{2} \left[\left(L_s i_{ds} + L_\mu i_{dr} \right) i_{qs} - \left(L_s i_{qs} + L_\mu i_{qr} \right) i_{ds} \right] = \frac{m p_p}{2} L_\mu \left(i_{dr} i_{qs} - i_{ds} i_{qr} \right) \tag{5}$$

The stator and rotor currents sum defines the magnetizing current

$$\boldsymbol{i_s} + \boldsymbol{i_r} = \boldsymbol{i_\mu} \tag{6}$$

The voltage of the transverse and rotor paths are equal, which is defined by the equivalent circuit parameters as

$$j\omega L_\mu \boldsymbol{i_\mu} = -\left(\frac{R_r}{s} + j\omega L_{r\sigma}\right)\boldsymbol{i_r} \tag{7}$$

The stator current is then defined by substitution of (7) and (2) to (6) as

$$\boldsymbol{i_s} = \boldsymbol{i_\mu} - \boldsymbol{i_r} = -\frac{\boldsymbol{i_r}}{j\omega L_\mu}\left(\frac{R_r}{s} + j\omega L_{r\sigma}\right) - \boldsymbol{i_r} = -\boldsymbol{i_r}\left(\frac{L_r}{L_\mu} - j\frac{R_r}{s\omega L_\mu}\right) \tag{8}$$

The rotor current is then

$$\boldsymbol{i_r} = -\left(\frac{L_r}{L_\mu} - j\frac{R_r}{\omega L_\mu s}\right)^{-1}\boldsymbol{i_s} = \frac{s\omega L_\mu}{-s\omega L_r + jR_r}\boldsymbol{i_s} = \frac{\omega_r L_\mu}{-\omega_r L_r + jR_r}\boldsymbol{i_s} = \boldsymbol{k_{ir}}\boldsymbol{i_s} \tag{9}$$

where ω_r is the rotor (slip) angular frequency. Coefficient k_{ir} can be decomposed to the real and imaginary part as

$$\boldsymbol{k_{ir}} = \frac{\omega_r L_\mu}{-\omega_r L_r + jR_r} = \frac{\omega_r L_\mu}{-\omega_r L_r + jR_r}\frac{-\omega_r L_r - jR_r}{-\omega_r L_r - jR_r} = \frac{-\omega_r^2 L_\mu L_r}{R_r^2 + \omega_r^2 L_r^2} + j\frac{-\omega_r L_\mu R_r}{R_r^2 + \omega_r^2 L_r^2} \tag{10}$$

and the rotor current components are

$$\begin{aligned} i_{dr} &= \frac{-\omega_r^2 L_\mu L_r}{R_r^2 + \omega_r^2 L_r^2} i_{ds} + \frac{\omega_r L_\mu R_r}{R_r^2 + \omega_r^2 L_r^2} i_{qs} = \frac{\omega_r L_\mu}{R_r^2 + \omega_r^2 L_r^2}\left(-\omega_r L_r i_{ds} + R_r i_{qs}\right) \\ i_{qr} &= \frac{-\omega_r L_\mu R_r}{R_r^2 + \omega_r^2 L_r^2} i_{ds} - \frac{\omega_r^2 L_\mu L_r}{R_r^2 + \omega_r^2 L_r^2} i_{qs} = \frac{\omega_r L_\mu}{R_r^2 + \omega_r^2 L_r^2}\left(-R_r i_{ds} - \omega_r L_r i_{qs}\right) \end{aligned} \tag{11}$$

The torque equation (5) is then modified to

$$\begin{aligned} T = \frac{m p_p}{2}\frac{\omega_r L_\mu^2}{R_r^2 + \omega_r^2 L_r^2}\left(\left(-\omega_r L_r i_{ds} + R_r i_{qs}\right) i_{qs} - i_{ds}\left(-R_r i_{ds} - \omega_r L_r i_{qs}\right)\right) = \\ = \frac{m p_p}{2}\frac{\omega_r L_\mu^2 R_r}{R_r^2 + \omega_r^2 L_r^2}\left(i_{ds}^2 + i_{qs}^2\right) = \frac{m p_p}{2}\frac{\omega_r L_\mu^2 R_r}{R_r^2 + \omega_r^2 L_r^2} I_m^2 \end{aligned} \tag{12}$$

It is proven, that for defined current amplitude, its distribution to the d-q reference system **does not affect the torque** because the rotor is isotropic and the d-q reference frame position is freely choosable. Stator voltage is generally defined as

$$\boldsymbol{v_s} = R_s\boldsymbol{i_s} + j\omega\boldsymbol{\Psi_s} = R_s\boldsymbol{i_s} + j\omega\left(L_s\boldsymbol{i_s} + L_\mu\boldsymbol{i_r}\right) \tag{13}$$

Substituting the rotor current from (9) and (10) and decomposition to the real and imaginary part (d-q components) gives

$$\begin{aligned} v_{ds} &= \left(R_s + \omega\frac{\omega_r L_\mu^2 R_r}{R_r^2+\omega_r^2 L_r^2}\right) i_{ds} - \omega\left(L_s - \frac{\omega_r^2 L_\mu^2 L_r}{R_r^2+\omega_r^2 L_r^2}\right) i_{qs} \\ v_{qs} &= \omega\left(L_s - \frac{\omega_r^2 L_\mu^2 L_r}{R_r^2+\omega_r^2 L_r^2}\right) i_{ds} + \left(R_s + \omega\frac{\omega_r L_\mu^2 R_r}{R_r^2+\omega_r^2 L_r^2}\right) i_{qs} \end{aligned} \tag{14}$$

The stator voltage amplitude is constant in the field weakening areas and is defined and modified to

$$V_m^2 = v_{ds}^2 + v_{qs}^2 = \left(i_{ds}^2 + i_{qs}^2\right)\left[\left(R_s + \omega\frac{\omega_r L_\mu^2 R_r}{R_r^2 + \omega_r^2 L_r^2}\right)^2 + \omega^2\left(L_s - \frac{\omega_r^2 L_\mu^2 L_r}{R_r^2 + \omega_r^2 L_r^2}\right)^2\right] \tag{15}$$

which is again independent of the distribution of the currents to the d-q reference system.

2.1 Matrix form

The above-derived equations can be also written in the matrix form known from the theory of optimal control of synchronous machines [2]. The torque is defined as

$$T = \frac{mp_p}{2}\left(\boldsymbol{i_s}\right)^\top \boldsymbol{J}\boldsymbol{\Psi_s} = \frac{mp_p}{2}\left(\boldsymbol{i_s}\right)^\top \boldsymbol{J}\left(\boldsymbol{L_s i_s} + \boldsymbol{L_\mu i_r}\right) = \frac{mp_p}{2}\left(\boldsymbol{i_s}\right)^\top \boldsymbol{J L_\mu i_r} \tag{16}$$

where all the equivalent circuit parameters must be written in the matrix form similar as the magnetizing inductance

$$\boldsymbol{L_\mu} = \begin{bmatrix} L_\mu & 0 \\ 0 & L_\mu \end{bmatrix} \tag{17}$$

and

$$\boldsymbol{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{18}$$

The equality of voltage of the transverse and rotor paths (7) is written as

$$\omega \boldsymbol{J L_\mu i_\mu} = -\left(\frac{\boldsymbol{R_r}}{s} + \omega \boldsymbol{J L_{r\sigma}}\right)\boldsymbol{i_r} \tag{19}$$

The stator current is

$$\boldsymbol{i_s} = \boldsymbol{i_\mu} - \boldsymbol{i_r} = -\left(\omega \boldsymbol{J L_\mu}\right)^{-1}\left(\frac{\boldsymbol{R_r}}{s} + \omega \boldsymbol{J L_{r\sigma}}\right)\boldsymbol{i_r} - \boldsymbol{i_r} = -\left(\boldsymbol{L_r}\left(\boldsymbol{L_\mu}\right)^{-1} + \boldsymbol{R_r}\left(s\omega \boldsymbol{J L_\mu}\right)^{-1}\right)\boldsymbol{i_r} \tag{20}$$

where

$$\boldsymbol{L_r}\left(\boldsymbol{L_\mu}\right)^{-1} = \left(\boldsymbol{L_{r\sigma}} + \boldsymbol{L_\mu}\right)\left(\boldsymbol{L_\mu}\right)^{-1} = \boldsymbol{L_{r\sigma}}\left(\boldsymbol{L_\mu}\right)^{-1} + \boldsymbol{I_2} \tag{21}$$

where $\boldsymbol{I_2}$ is 2×2 identity matrix. The rotor current is then

$$\boldsymbol{i_r} = -\left(\boldsymbol{L_r}\left(\boldsymbol{L_\mu}\right)^{-1} + \boldsymbol{R_r}\left(\omega_r \boldsymbol{J}\boldsymbol{L_\mu}\right)^{-1}\right)^{-1}\boldsymbol{i_s} = \boldsymbol{k_{ir}}\boldsymbol{i_s} \tag{22}$$

where

$$\boldsymbol{k_{ir}} = \frac{1}{R_r^2 + \omega_r^2 L_r^2}\begin{bmatrix} -\omega_r^2 L_r L_\mu & \omega_r R_r L_\mu \\ -\omega_r R_r L_\mu & -\omega_r^2 L_r L_\mu \end{bmatrix} \tag{23}$$

The matrix form of the torque equation is then modified to

$$T = \frac{m p_p}{2}\left(\boldsymbol{i_s}\right)^\top \boldsymbol{J}\boldsymbol{L_\mu}\boldsymbol{k_{ir}}\boldsymbol{i_s} \tag{24}$$

The stator voltage equation is generally defined in the matrix form as

$$\boldsymbol{v_s} = R_s\boldsymbol{i_s} + \omega\boldsymbol{J}\boldsymbol{\Psi_s} = R_s\boldsymbol{i_s} + \omega\boldsymbol{J}\left(\boldsymbol{L_s}\boldsymbol{i_s} + \boldsymbol{L_\mu}\boldsymbol{i_r}\right) = R_s\boldsymbol{i_s} + \omega\boldsymbol{J}\left(\boldsymbol{L_s} + \boldsymbol{L_\mu}\boldsymbol{k_{ir}}\right)\boldsymbol{i_s} \tag{25}$$

2.2 IM optimal control

Since the torque and voltage equations are independent of the current decomposition to the d-q reference frame, the only parameter in the torque and voltage equation is the rotor angular frequency and the slip of the rotor.

The torque equation (12) derivative with respect to ω_r will be set to equal to zero as

$$\frac{\partial T}{\partial \omega_r} = \frac{m p_p}{2}\frac{L_\mu^2 R_r\left(R_r^2 + \omega_r^2 L_r^2\right) - 2\omega_r^2 L_r^2 L_\mu^2 R_r}{\left(R_r^2 + \omega_r^2 L_r^2\right)^2} I_m^2 = 0 \tag{26}$$

and the optimal ω_r is calculated as

$$\omega_r = \pm\frac{R_r}{L_r} \tag{27}$$

for both positive and negative torques which implies the MTPA definition (for constant equivalent circuit parameters). Generally, in the field weakening area, the rotor frequency must be increased to meet the voltage limit requirements. Equivalently, other regimes could be derived for stator current vector placed generally in the d-q reference frame. In practical applications, the rotor flux-oriented control is often applied and the flux linkages and currents are connected to the d-q reference system which makes the calculations easier. Therefore, this control will be expected now.

2.3 Rotor flux oriented control

The rotor flux-oriented control is defined as $\Psi_{qr} = 0$. This flux linkage is by (1) defined as

$$\Psi_{qr} = L_\mu i_{qs} + L_r i_{qr} = 0 \tag{28}$$

and the rotor quadrature-axis current is

$$i_{qr} = -\frac{L_\mu}{L_r} i_{qs} \tag{29}$$

Decomposition of the stator current to d-q reference frame leads to

$$\begin{aligned} i_{ds} &= -i_{dr}\tfrac{L_r}{L_\mu} - i_{qr}\tfrac{R_r}{\omega_r L_\mu} \\ i_{qs} &= i_{dr}\tfrac{R_r}{\omega_r L_\mu} - i_{qr}\tfrac{L_r}{L_\mu} \end{aligned} \tag{30}$$

Substitution of (29) to (30) defines

$$\begin{aligned} i_{ds} &= -i_{dr}\tfrac{L_r}{L_\mu} + \tfrac{R_r}{\omega_r L_r} i_{qs} \\ i_{qs} &= i_{dr}\tfrac{R_r}{\omega_r L_\mu} + i_{qs} \end{aligned} \tag{31}$$

The second equation of (31) is valid only when $i_{dr} = 0$. Equation (31) is then

$$\begin{aligned} i_{ds} &= -i_{qr}\tfrac{R_r}{\omega_r L_\mu} \\ i_{qs} &= -i_{qr}\tfrac{L_r}{L_\mu} \end{aligned} \tag{32}$$

Comparing the i_{qr} currents in (32), we get general expression for ω_r valid for every operation point as

$$\omega_r = \frac{R_r}{L_r}\frac{i_{qs}}{i_{ds}} \tag{33}$$

The k_{ir} coefficient (10) is then

$$\boldsymbol{k_{ir}} = \frac{-\left(\frac{R_r}{L_r}\frac{i_{qs}}{i_{ds}}\right)^2 L_\mu L_r}{R_r^2 + \left(\frac{R_r}{L_r}\frac{i_{qs}}{i_{ds}}\right)^2 L_r^2} + j\frac{-\frac{R_r}{L_r}\frac{i_{qs}}{i_{ds}} L_\mu R_r}{R_r^2 + \left(\frac{R_r}{L_r}\frac{i_{qs}}{i_{ds}}\right)^2 L_r^2} = -\frac{L_\mu}{L_r}\frac{i_{qs}^2}{i_{ds}^2 + i_{qs}^2} - j\frac{L_\mu}{L_r}\frac{i_{ds} i_{qs}}{i_{ds}^2 + i_{qs}^2} \tag{34}$$

Substituting second equation of (32) to (5) while $i_{dr} = 0$ leads to

$$T = \frac{mp_p}{2}\frac{L_\mu^2}{L_r} i_{ds} i_{qs} = k_T i_{ds} i_{qs} \tag{35}$$

The stator voltage can be obtained by substitution of (33) into (14) as

$$\begin{aligned} v_{ds} &= \left(R_s + \omega \frac{\frac{L_\mu^2}{L_r} i_{ds} i_{qs}}{i_{ds}^2 + i_{qs}^2} \right) i_{ds} - \omega \left(L_s - \frac{\frac{L_\mu^2}{L_r} i_{qs}^2}{i_{ds}^2 + i_{qs}^2} \right) i_{qs} = R_s i_{ds} - \omega L_s' i_{qs} \\ v_{qs} &= \omega \left(L_s - \frac{\frac{L_\mu^2}{L_r} i_{qs}^2}{i_{ds}^2 + i_{qs}^2} \right) i_{ds} + \left(R_s + \omega \frac{\frac{L_\mu^2}{L_r} i_{ds} i_{qs}}{i_{ds}^2 + i_{qs}^2} \right) i_{qs} = \omega L_s i_{ds} + R_s i_{qs} \end{aligned} \tag{36}$$

or simplier by substitution of rotor current $i_r = 0 - j\frac{L_\mu}{L_r} i_{qs}$ into (13). Inductance $L_s' = L_s - \frac{L_\mu^2}{L_r}$ only helps to simplify the equations.

2.3.1 Matrix form

Under these conditions, the stator flux linkage in the matrix form is defined as

$$\boldsymbol{\Psi_s} = L_s \begin{bmatrix} i_{ds} \\ i_{qs} \end{bmatrix} + L_\mu \begin{bmatrix} 0 \\ -\frac{L_\mu}{L_r} i_{qs} \end{bmatrix} = \begin{bmatrix} L_s & 0 \\ 0 & L_s' \end{bmatrix} \begin{bmatrix} i_{ds} \\ i_{qs} \end{bmatrix} = \boldsymbol{L_s i_s} \tag{37}$$

and the voltage

$$\boldsymbol{v_s} = R_s \boldsymbol{i_s} + \omega \boldsymbol{J L_s i_s} \tag{38}$$

The torque is then defined from (16) as

$$T = \frac{mp_p}{2} \boldsymbol{i}_s^\top \boldsymbol{J \Psi_s} = \frac{mp_p}{2} \boldsymbol{i}_s^\top \boldsymbol{J L_s i_s} \tag{39}$$

The symmetrical matrix A is defined by [2] as

$$\boldsymbol{A} = \frac{mp_p}{4} \left(\boldsymbol{J L_s} + \boldsymbol{L_s J}^\top \right) = \frac{mp_p}{2} \begin{bmatrix} 0 & \frac{L_\mu^2}{2L_r} \\ \frac{L_\mu^2}{2L_r} & 0 \end{bmatrix} = \boldsymbol{A}^\top \tag{40}$$

and the torque is then

$$T = (\boldsymbol{i_s})^\top \boldsymbol{A i_s} \tag{41}$$

Now the general equation describing the IM are known and specific operation areas and point will be discussed.

2.4 Maximum torque per Ampére (MTPA)

Presuming linear magnetic circuit and constant equivalent circuit parameters, the optimal control of the induction machine results in $i_{ds} = i_{qs}$ according to (35). The optimal rotor frequency is defined in (27) as $\omega_r = \pm\frac{R_r}{L_r}$ for both positive and negative torques. The maximal torque is limited by the condition

$$i_{ds}^2 + i_{qs}^2 \leq I_{max}^2 \tag{42}$$

2.4.1 Maximal speed at MTPA

For every MTPA point, the stator currents are known. The voltage is limited by

$$v_{ds}^2 + v_{qs}^2 \leq V_{max}^2 \tag{43}$$

From (36) and (43), we get equation

$$\left(R_s i_{ds} - \omega L_s' i_{qs}\right)^2 + \left(R_s i_{qs} + \omega L_s i_{ds}\right)^2 = V_{max}^2 \tag{44}$$

which leads to

$$\left(L_s^2 i_{ds}^2 + \left(L_s'\right)^2 i_{qs}^2\right)\omega^2 + 2R_s i_{ds} i_{qs}\left(L_s - L_s'\right)\omega + R_s^2\left(i_{ds}^2 + i_{qs}^2\right) - V_{max}^2 = 0 \tag{45}$$

and the maximum MTPA speed is

$$\omega = \frac{-R_s \frac{L_\mu^2}{L_r} i_{ds} i_{qs} \pm \sqrt{R_s^2 \frac{L_\mu^4}{L_r^2} i_{ds}^2 i_{qs}^2 - \left(L_s^2 i_{ds}^2 + \left(L_s'\right)^2 i_{qs}^2\right)\left(R_s^2\left(i_{ds}^2 + i_{qs}^2\right) - V_{max}^2\right)}}{L_s^2 i_{ds}^2 + \left(L_s'\right)^2 i_{qs}^2} \tag{46}$$

This limit can be calculated for every torque value.

2.5 Maximal Torque Characteristics

A different approach is applied when the maximal torque-speed characteristics are calculated, limited by (42) and (43). With increasing speed, the optimal torque is reached by MTPA until limit (43) is reached at speed defined by (46). For higher speeds, maximum Current (MC) and maximum torque per volt (MTPV) are applied.

2.5.1 Maximum Current (MC)

Maximal torque at speeds higher than nominal speed is obtained by

$$\max T \text{ s.t. } \begin{aligned} i_{ds}^2 + i_{qs}^2 &= I_{max}^2 \\ v_{ds}^2 + v_{qs}^2 &= V_{max}^2 \end{aligned} \tag{47}$$

Expressing i_{qs} as

$$i_{qs} = \sqrt{I_{max}^2 - i_{ds}^2}, \tag{48}$$

equation (44) is then

$$2R_s\omega\left(L_s - L_s'\right) i_{ds}\sqrt{I_{max}^2 - i_{ds}^2} = V_{max}^2 - \left(R_s^2 + \omega^2\left(L_s'\right)^2\right) I_{max}^2 + \omega^2\left(\left(L_s'\right)^2 - L_s^2\right) i_{ds}^2 \tag{49}$$

Current i_{ds} is then equal to

$$i_{ds}^2 = \frac{-B - \sqrt{B^2 - 4AC}}{2A} \tag{50}$$

where

$$\begin{gathered} A = \omega^2\left(L_s' - L_s\right)^2\left[\omega^2\left(L_s' + L_s\right)^2 + 4R_s^2\right] \\ B = 2D\omega^2\left(\left(L_s'\right)^2 - L_s^2\right) - 4R_s^2\omega^2\left(L_s' - L_s\right)^2 I_{max}^2 \\ C = D^2 \\ D = V_{max}^2 - \left(R_s^2 + \omega^2\left(L_s'\right)^2\right) I_{max}^2 \end{gathered} \tag{51}$$

2.5.2 maximum Torque per Volt (MTPV)

For higher speeds, the machine operates at the voltage limit whereas the current can operate under its limit. The stator currents are derived from (36) as

$$\begin{aligned} i_{ds} &= \tfrac{R_s v_{ds} + \omega L_s' v_{qs}}{R_s^2 + \omega^2 L_s' L_s} \\ i_{qs} &= \tfrac{-\omega L_s v_{ds} + R_s v_{qs}}{R_s^2 + \omega^2 L_s' L_s} \end{aligned} \tag{52}$$

The torque is then

$$T = \frac{k_T}{R_s^2 + \omega^2 L_s' L_s}\left[R_s\omega\left(-L_s v_{ds}^2 + L_s' v_{qs}^2\right) + \left(R_s^2 - \omega^2 L_s' L_s\right) v_{ds} v_{qs}\right] \tag{53}$$

and it is maximized at constraints

$$\max T \text{ s.t. } v_{ds}^2 + v_{qs}^2 = V_{max}^2 \tag{54}$$

The d-axis component of voltage is

$$v_{ds} = \pm\sqrt{V_{max}^2 - v_{qs}^2} \tag{55}$$

and the torque is

$$T = k_T \frac{R_s\omega\left(-L_s\left(V_{max}^2 - v_{qs}^2\right) + L'_s v_{qs}^2\right) \pm \left(R_s^2 - \omega^2 L'_s L_s\right) v_{qs}\sqrt{V_{max}^2 - v_{qs}^2}}{R_s^2 + \omega^2 L'_s L_s} \tag{56}$$

Voltage v_{ds} for maximum torque is obtained by the torque derivative equals zero simplified to

$$\frac{dT}{dv_{ds}} = 2R_s\omega\left(L_s + L'_s\right) v_{qs} \pm \left(R_s^2 - \omega^2 L'_s L_s\right)\sqrt{V_{max}^2 - v_{qs}^2} \mp \frac{\left(R_s^2 - \omega^2 L'_s L_s\right)}{\sqrt{V_{max}^2 - v_{ds}^2}} v_{qs}^2 = 0 \tag{57}$$

Which implies in the quadratic equation of v_{qs}^2 with solution

$$v_{qs}^2 = \frac{V_{max}^2}{2} \pm \frac{V_{max}^2 R_s\omega\left(L_s + L'_s\right)\sqrt{\left(R_s^2 + \omega^2 L_s^2\right)\left(R_s^2 + \omega^2\left(L'_s\right)^2\right)}}{2\left(R_s^2 + \omega^2 L_s^2\right)\left(R_s^2 + \omega^2\left(L'_s\right)^2\right)} \tag{58}$$

valid for the plus sign (equivalently minus sign is valid for v_{ds}^2). The minimum speed of MTPV is defined as a point where the current limit I_{max} under the MTPV control is reached.

2.6 Field Weakening (FW) Area

It is a big area of IM control in higher speeds where MTPA cannot be reached and we are not at the maximum torque curve. The stator voltage is defined by (44). For chosen machine torque, expressing i_{ds} from (35) and substituting into voltage equation gives

$$i_{qs}^4\left(R_s^2 + \omega^2\left(L'_s\right)^2\right) + i_{qs}^2\left[2\omega R_s\frac{T}{k_T}\left(L_s - L'_s\right) - V_{max}^2\right] + \frac{T^2}{k_T^2}\left(R_s^2 + \omega^2 L_s^2\right) = 0 \tag{59}$$

Current i_{qs} is calculated and i_{ds} is obtained from (35).

3 First + Third harmonic Control

The idea of first+third harmonic control of IM is based on presented papers relating to the multiphase machines and nine-phase IM [3] and the paper presenting the solution of optimal currents for five-phase PMSM [1]. The nomenclature of different operation areas and regimes is taken from [1]. The solution is based on the matrix expression of the torque and voltage equations and the numerical solution of the optimization problem. Therefore, this part will be mainly focused on the specific problematics of IM and its practical solution. The parameters of the equivalent circuit are enlarged to the third-harmonic parameters and distinguished by superscripts ^{1} and ^{3}. The third harmonic injection is seen as the creation of two virtual machines coupled through the same synchronous speed of the magnetic field n_s and the mechanical speed of the shaft. From the slip definition

$$s = \frac{n_s - n}{n_s} \tag{60}$$

implies that the slip is constant for both virtual machines. The synchronous angular frequency of the third harmonic machine is three times the synchronous angular frequency of the first harmonic machine

$$\omega^3 = 3\omega^1, \tag{61}$$

just as the rotor frequency

$$\begin{aligned} \omega_r^1 &= \omega_r = s\omega^1 \\ \omega_r^3 &= s\omega^3 = 3\omega_r^1 \end{aligned} \tag{62}$$

For every virtual machine, the optimal rotor frequency is given by the equivalent circuit parameters and stator currents. Therefore, the situation with both machines optimally controlled is generally impossible; it can occur only for specific conditions or at specific operation points. For simplification, let us expect at first look that both machines are controlled optimally in the MTPA.

3.1 MTPA if both virtual machines are controlled optimally

The torque produced by every harmonic component ν is

$$T^\nu = \frac{\nu m p_p}{2} \frac{\left(L_\mu^\nu\right)^2}{L_r^\nu} i_{ds}^\nu i_{qs}^\nu \tag{63}$$

Expecting optimal control, the total torque is

$$T = T^1 + T^3 = \frac{m p_p}{2} \frac{\left(L_\mu^1\right)^2}{L_r^1} \frac{\left(I_m^1\right)^2}{2} + \frac{3 m p_p}{2} \frac{\left(L_\mu^3\right)^2}{L_r^3} \frac{\left(I_m^3\right)^2}{2} \tag{64}$$

The equivalent current amplitude I_m is defined as

$$(I_m)^2 = \left(I_m^1\right)^2 + \left(I_m^3\right)^2 \tag{65}$$

The amplitude of the first harmonic current can be defined as

$$\left(I_m^1\right)^2 = (I_m)^2 \left(1 - \left(i_{pu}^3\right)^2\right) \tag{66}$$

where $i_{pu}^3 = I_m^3/I_m$ is the third harmonic current amplitude in the per unit system. The torque is then

$$T = \frac{mp_p}{4} \frac{\left(L_\mu^1\right)^2}{L_r^1} (I_m)^2 + \frac{mp_p}{4} \left(3\frac{\left(L_\mu^3\right)^2}{L_r^3} - \frac{\left(L_\mu^1\right)^2}{L_r^1}\right) (I_m)^2 \left(i_{pu}^3\right)^2 \tag{67}$$

The first part of the equation represents the torque produced if only the first harmonic current is applied. The torque can be generally maximized under three conditions:

1. $3\frac{\left(L_\mu^3\right)^2}{L_r^3} < \frac{\left(L_\mu^1\right)^2}{L_r^1}$
   In this case, the second part of the torque equation is negative and the torque is maximized if $i_{pu}^3 = 0$. The torque equation is then (generalized back to d-q currents)

$$T = \frac{mp_p}{2} \frac{\left(L_\mu^1\right)^2}{L_r^1} i_{ds}^1 i_{qs}^1 \tag{68}$$

2. $3\frac{\left(L_\mu^3\right)^2}{L_r^3} > \frac{\left(L_\mu^1\right)^2}{L_r^1}$
   The second part of the torque equation is positive and the torque is maximized when $i_{pu}^3 = 1$. The torque is then modified to

$$T = \frac{3mp_p}{2} \frac{\left(L_\mu^3\right)^2}{L_r^3} i_{ds}^3 i_{qs}^3 \tag{69}$$

3. $3\frac{\left(L_\mu^3\right)^2}{L_r^3} = \frac{\left(L_\mu^1\right)^2}{L_r^1}$
   The second part of the torque equation is zero and the torque can be maximized by using both only first harmonic or only third harmonic currents.

3.1.1 Matrix form

The induction machine parameters are extended to the four-dimensional matrices as e.g. the stator inductance matrix

$$\boldsymbol{L_s} = \begin{bmatrix} L_s^1 & 0 & 0 & 0 \\ 0 & L_s^1 & 0 & 0 \\ 0 & 0 & L_s^3 & 0 \\ 0 & 0 & 0 & L_s^3 \end{bmatrix} \tag{70}$$

The matrix $\boldsymbol{J}$ is

$$\boldsymbol{J} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \end{bmatrix} \tag{71}$$

and the current vectors are $\boldsymbol{i_s} = [i_{sd}^1, i_{sq}^1, i_{sd}^3, i_{sq}^3]^\top$ and $\boldsymbol{i_r} = [i_{rd}^1, i_{rq}^1, i_{rd}^3, i_{rq}^3]^\top$. If both machines are controlled optimally ($\Psi_{qr}^1 = \Psi_{qr}^3 = 0$), the rotor current vector is modified to $\boldsymbol{i_r} = [0, -\frac{L_\mu^1}{L_r^1} i_{qs}^1, 0, -\frac{L_\mu^3}{L_r^3} i_{qs}^3]^\top$ The stator flux linkage is then

$$\begin{aligned} \boldsymbol{\Psi_s} = \boldsymbol{L_s i_s} + \boldsymbol{L_\mu i_r} = \begin{bmatrix} L_s^1 & 0 & 0 & 0 \\ 0 & L_s^1 & 0 & 0 \\ 0 & 0 & L_s^3 & 0 \\ 0 & 0 & 0 & L_s^3 \end{bmatrix} \begin{bmatrix} i_{sd}^1 \\ i_{sq}^1 \\ i_{sd}^3 \\ i_{sq}^3 \end{bmatrix} + \begin{bmatrix} L_\mu^1 & 0 & 0 & 0 \\ 0 & L_\mu^1 & 0 & 0 \\ 0 & 0 & L_\mu^3 & 0 \\ 0 & 0 & 0 & L_\mu^3 \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{L_\mu^1}{L_r^1} i_{qs}^1 \\ 0 \\ -\frac{L_\mu^3}{L_r^3} i_{qs}^3 \end{bmatrix} = \\ = \begin{bmatrix} L_s^1 & 0 & 0 & 0 \\ 0 & L_s^1 - \frac{\left(L_\mu^1\right)^2}{L_r^1} & 0 & 0 \\ 0 & 0 & L_s^3 & 0 \\ 0 & 0 & 0 & L_s^3 - \frac{\left(L_\mu^3\right)^2}{L_r^3} \end{bmatrix} \begin{bmatrix} i_{sd}^1 \\ i_{sq}^1 \\ i_{sd}^3 \\ i_{sq}^3 \end{bmatrix} \end{aligned} \tag{72}$$

The symmetrical matrix $\boldsymbol{A}$ is according to (40)

$$\boldsymbol{A} = \frac{m p_p}{2} \begin{bmatrix} 0 & \frac{L_\mu^1}{2L_r^1} & 0 & 0 \\ \frac{L_\mu^1}{2L_r^1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 3\frac{L_\mu^3}{2L_r^3} \\ 0 & 0 & 3\frac{L_\mu^3}{2L_r^3} & 0 \end{bmatrix} \tag{73}$$

The optimization problem can be then built as

$$
\begin{aligned}
\min \ & \boldsymbol{i_s}^\top \boldsymbol{i_s} \\
\text{s.t. } & \boldsymbol{i_s}^\top \boldsymbol{A} \boldsymbol{i_s} - T = 0
\end{aligned}
\tag{74}
$$

The Lagrangian is formulated as

$$
\mathcal{L}\left(\boldsymbol{i_s}, \lambda\right) = -\boldsymbol{i_s}^\top \boldsymbol{i_s} + \lambda \left(\boldsymbol{i_s}^\top \boldsymbol{A} \boldsymbol{i_s} - T\right) \tag{75}
$$

The derivative by $\boldsymbol{i_s}$ must be equal to zero

$$
\frac{d\mathcal{L}\left(\boldsymbol{i_s}, \lambda\right)}{d\boldsymbol{i_s}} = -\boldsymbol{i_s} + \lambda \boldsymbol{A} \boldsymbol{i_s} = 0 \tag{76}
$$

which is

$$
-\begin{bmatrix} i_{sd}^1 \\ i_{sq}^1 \\ i_{sd}^3 \\ i_{sq}^3 \end{bmatrix} + \lambda \frac{m p_p}{2} \begin{bmatrix} 0 & \frac{L_\mu^1}{2L_r^1} & 0 & 0 \\ \frac{L_\mu^1}{2L_r^1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 3\frac{L_\mu^3}{2L_r^3} \\ 0 & 0 & 3\frac{L_\mu^3}{2L_r^3} & 0 \end{bmatrix} \begin{bmatrix} i_{sd}^1 \\ i_{sq}^1 \\ i_{sd}^3 \\ i_{sq}^3 \end{bmatrix} = 0 \tag{77}
$$

Since λ is not a matrix, this set of equations can be valid only for $\left|i_{sd}^1\right| = \left|i_{sq}^1\right|$ and $\left|i_{sd}^3\right| = \left|i_{sq}^3\right|$ in two special cases:

1. $i_{sd}^3 = i_{sq}^3 = 0$ and $\lambda = \pm \frac{4}{m p_p} \frac{L_r^1}{L_\mu^1}$
2. $i_{sd}^1 = i_{sq}^1 = 0$ and $\lambda = \pm \frac{4}{3 m p_p} \frac{L_r^3}{L_\mu^3}$

which corresponds with the results from the previous chapter.

3.1.2 Real IM parameters

The magnetizing inductance for different harmonic components is

$$
L_\mu^\nu = \mu_0 \alpha_\delta m \frac{t_p}{\pi p \delta''} l_{Fe} \left(N_s \frac{k_{v\nu}}{\nu}\right)^2 \tag{78}
$$

Even if the winding factors of both harmonic components are equal (e.g. full step winding with $q = 1$ leads to $k_v = 1$ of every harmonic component), the magnetizing inductance is proportional to $1/\nu^2$. The rotor inductance of every component is defined by (2). The rotor leakage inductance $L_{r\sigma}$ almost does not change for different harmonic components and therefore the fraction $\left(L_\mu^3\right)^2 / L_r^3$ is more than 9-times smaller than $\left(L_\mu^1\right)^2 / L_r^1$. If the idealized rotor inductances are used $L_{r\sigma}^{\nu} = 0$, the expression in parentheses in (67) equals

$$3\frac{L_{\mu}^{3\,2}}{L_r^3} - \frac{L_{\mu}^{1\,2}}{L_r^1} = 3L_{\mu}^3 - L_{\mu}^1 = -\frac{2}{3}L_{\mu}^1 \tag{79}$$

Even if the most optimal solution is considered, the torque produced by the combination of the first and third harmonic components cannot exceed the first harmonic torque (if the voltage is not limited). Therefore, **in the MTPA operation, only the first harmonic currents are applied**.

In the real machine, both harmonic components will not operate with the optimal rotor frequency. Therefore, the torque equation (63) is not valid, and general equations must be used.

3.2 *Real control*

Only the matrix-based solution will be presented here. Equations (16), (19) - (22), and (24) - (25) are valid with the extended equivalent circuit components to 4 dimensions. The coefficient matrix $\boldsymbol{k_{ir}}$ equals

$$\boldsymbol{k_{ir}} = \begin{bmatrix} \frac{-\omega_r^2 L_\mu^1 L_r^1}{(R_r^1)^2+\omega_r^2(L_r^1)^2} & \frac{\omega_r L_\mu^1 R_r^1}{(R_r^1)^2+\omega_r^2(L_r^1)^2} & 0 & 0 \\ \frac{-\omega_r L_\mu^1 R_r^1}{(R_r^1)^2+\omega_r^2(L_r^1)^2} & \frac{-\omega_r^2 L_\mu^1 L_r^1}{(R_r^1)^2+\omega_r^2(L_r^1)^2} & 0 & 0 \\ 0 & 0 & \frac{-9\omega_r^2 L_\mu^3 L_r^3}{(R_r^3)^2+9\omega_r^2(L_r^3)^2} & \frac{3\omega_r L_\mu^3 R_r^3}{(R_r^3)^2+9\omega_r^2(L_r^3)^2} \\ 0 & 0 & \frac{-3\omega_r L_\mu^3 R_r^3}{(R_r^2)^2+9\omega_r^2(L_r^3)^2} & \frac{-9\omega_r^2 L_\mu^3 L_r^3}{(R_r^3)^2+9\omega_r^2(L_r^3)^2} \end{bmatrix} \tag{80}$$

The final matrix $\boldsymbol{A} = \frac{mp_p}{2}\boldsymbol{J}\boldsymbol{L_\mu}\boldsymbol{k_{ir}}$ can be obtained again and the matrix solution is used in the optimization process. The matrix is not symmetrical but it is OK with the used optimization. The problematic part of the optimization is that the only parameter that is changed and defines the operation point is the rotor angular frequency. Since the rotor angular frequency is included in the matrix $\boldsymbol{A}$, the matrix is changing and the solved problem is nonlinear. That complicates the calculation and therefore the matrix must be defined for every calculation separately and the optimal rotor frequency is found numerically. Only in the MTPA case, the currents are defined manually.

Tab. 4.1: IM equivalent circuit parameters

Parameter	Value		Unit
	First harmonic	**Third harmonic**	
Number of phases m	9		-
Number of polepairs p_p	2		-
Stator resistance R_s	1.36		Ω
Rotor resistance R_r	1.09	1.05	Ω
Magnetizing inductance L_μ	685	88.1	mH
Stator leakage inductance $L_{s\sigma}$	13.4	14.4	mH
Stator inductance L_s	698	101.5	mH
Rotor leakage inductance $L_{r\sigma}$	32.7	38.9	mH
Rotor inductance L_r	718	127.0	mH
Resistance respecting core loss R_{Fe}	2344	1224	Ω

4 Results

The calculations are verified on an example of a nine-phase induction machine with parameters summarized in Tab. 4.1. The resistance respecting core loos is neglected in the mathematical model but it is shown here for completeness. The voltage amplitude is limited to $V_m = 230\sqrt{2}$ V and the current amplitude to $I_m = 7.5$ A. Maximum speed is $n_{max} = 5000$ min^{-1}.

4.1 Maximum torque characteristic

The results of modeling only the first harmonic control and a combination of the first and third harmonic control are compared. Fig. 4.1 shows a comparison of the current trajectory in the d-q plane (both first and third harmonic currents). Comparing the first harmonic current of both control strategies, it can be seen that the trajectories in the MTPA (blue) and MTPV (grey) operations are identical for both control strategies, but the speed point where MTPV is reached is different, see below. The biggest difference is in comparison of the MC (yellow) operation of the first harmonic control and MTPA II (red) and MC&MT (yellow) of the first+third harmonic control where it is obvious the increase of the first harmonic current amplitude thanks to the third harmonic current.

The resulting torque-speed and power-speed characteristics are shown in 4.2 and 4.3, where dashed lines correspond with only the first harmonic control and solid lines with first+third harmonic control. The increase in the maximum torque and power is significant. The speed where the control switches to the MTPV also increases in the first+third harmonic control.
A comparison of the rotor angular frequency ω_r is shown in Fig. 4.4. The rotor frequency is

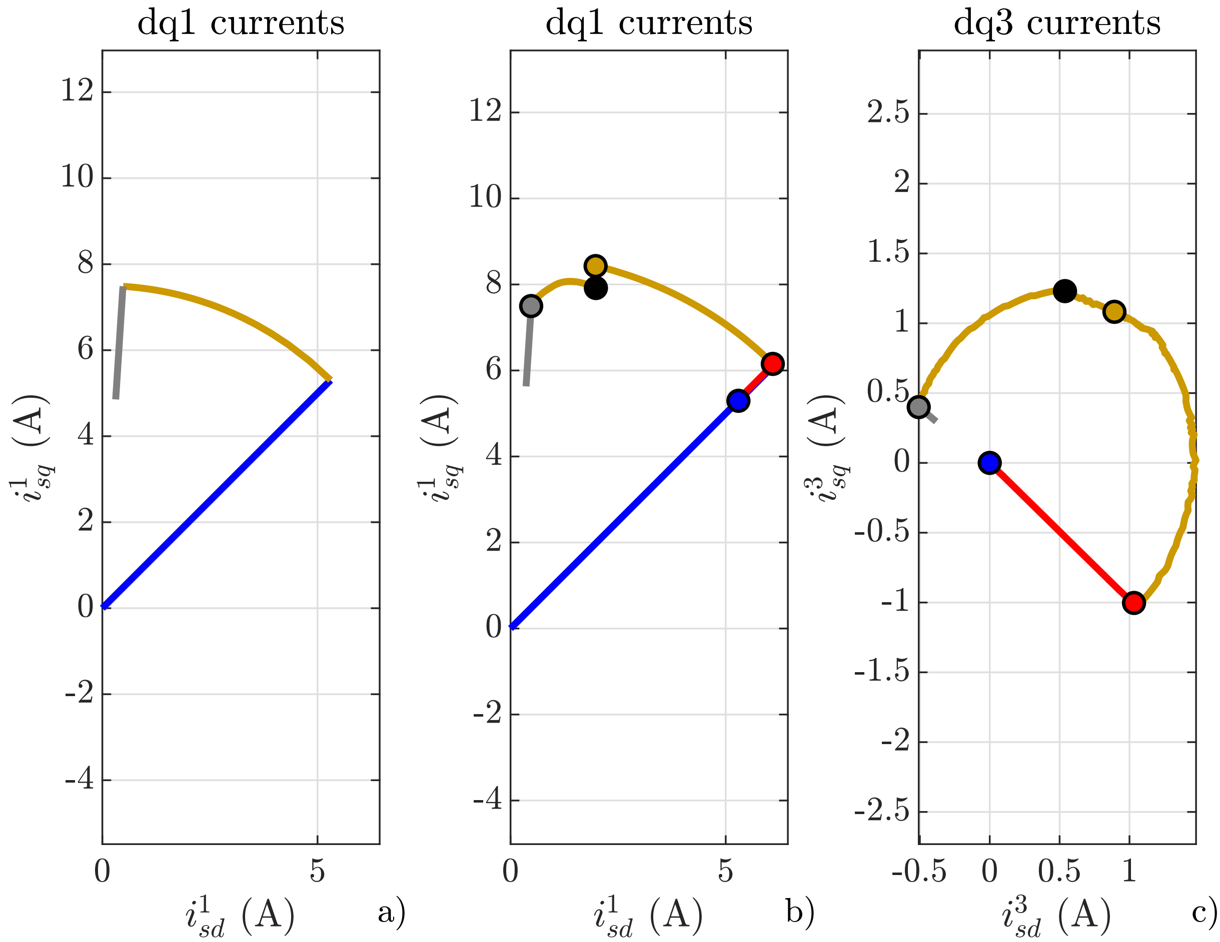

Fig. 4.1: Stator current in the complex plane (a) only first harmonic control, (b) first+third harmonic control - first harmonic current, (c) first+third harmonic control - third harmonic current

higher up to the yellow P3 point (the last point with a symmetrical current waveform). After that, the rotor frequency slightly decreases until the black P6 point is reached (the first point of symmetrical voltage waveform) and the rotor frequency then increases but it is smaller in comparison with the first harmonic control, which indicates smaller rotor currents.

The graphs of amplitude and phase shift of stator currents and voltages are shown in Fig. 4.5 and Fig. 4.6. The area of the decrease of the rotor frequency (between points P3 and P6) is also an area of decrease of first harmonic current and increase of the voltage.

The last figure (Fig. 4.7) displays a comparison of the Joule losses of the first and first+third harmonic control defined as

$$\Delta P_j = \frac{m}{2}\boldsymbol{i_s}^\top \boldsymbol{R_s}\boldsymbol{i_s} + \frac{m}{2}\boldsymbol{i_r}^\top \boldsymbol{R_r}\boldsymbol{i_r} \tag{81}$$

where $\boldsymbol{R_s}$ and $\boldsymbol{R_r}$ are defined similarly as $\boldsymbol{L_s}$ in (70) and i_r is defined from the equivalent

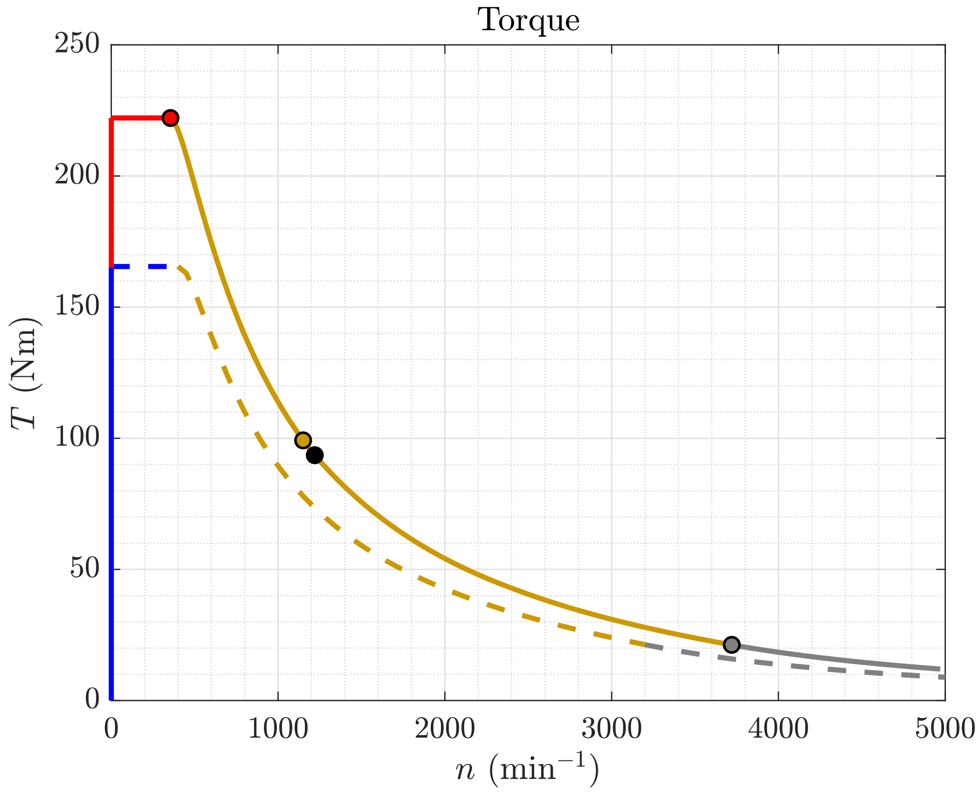

Fig. 4.2: Torque-speed characteristic of only first harmonic control (dashed lines) and first+third harmonic control (solid lines)

circuit or as $\boldsymbol{i}_r = \boldsymbol{k}_{ir}\boldsymbol{i}_s$, where $\boldsymbol{k}_{ir}$ is defined in (80).

It is obvious that the decrease in the Joule losses in the flux weakening is significant when the third harmonic is added.

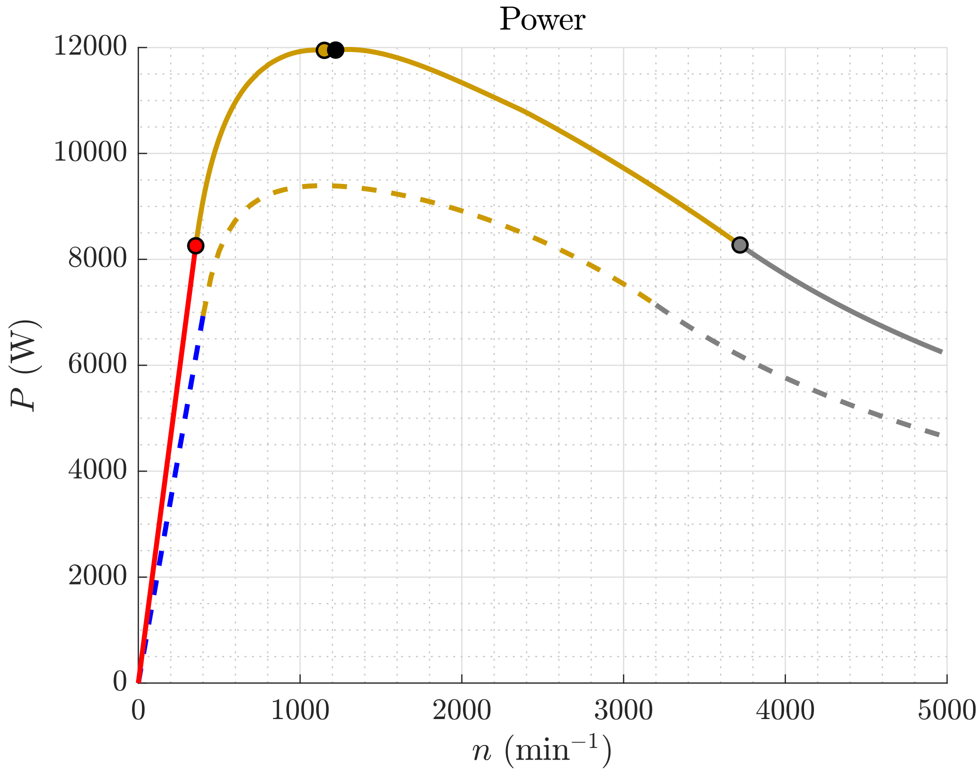

Fig. 4.3: Power-speed characteristic of only first harmonic control (dashed lines) and first+third harmonic control (solid lines)

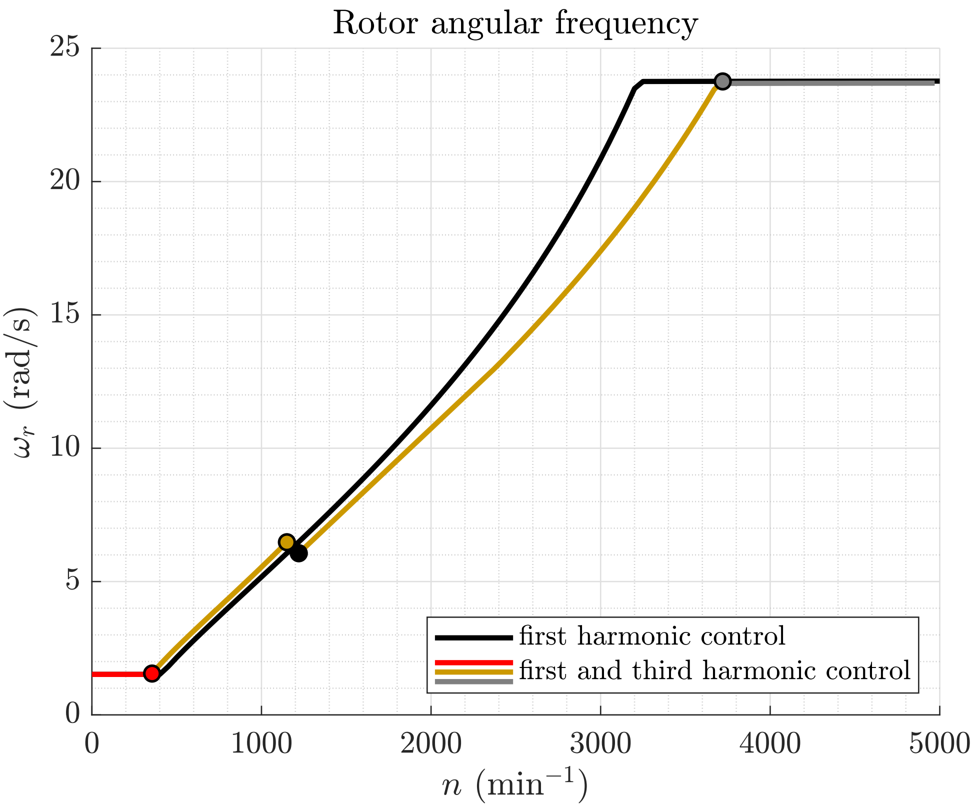

Fig. 4.4: Rotor angular frequency of the maximum torque curve

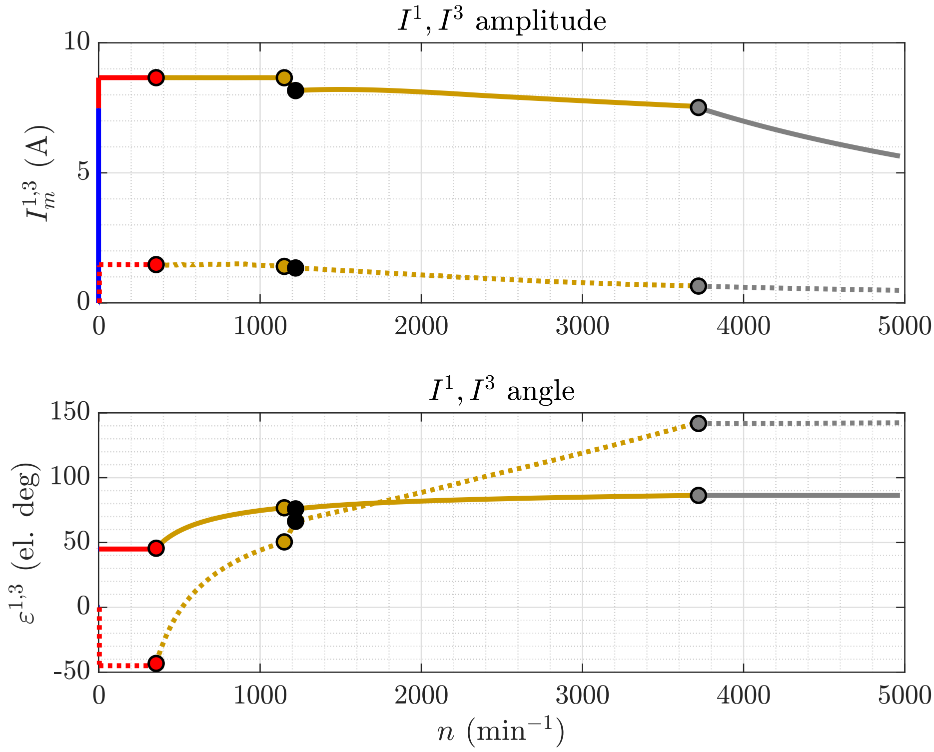

Fig. 4.5: Current amplitude and angle of first harmonic (solid line) and third harmonic (dashed line) component

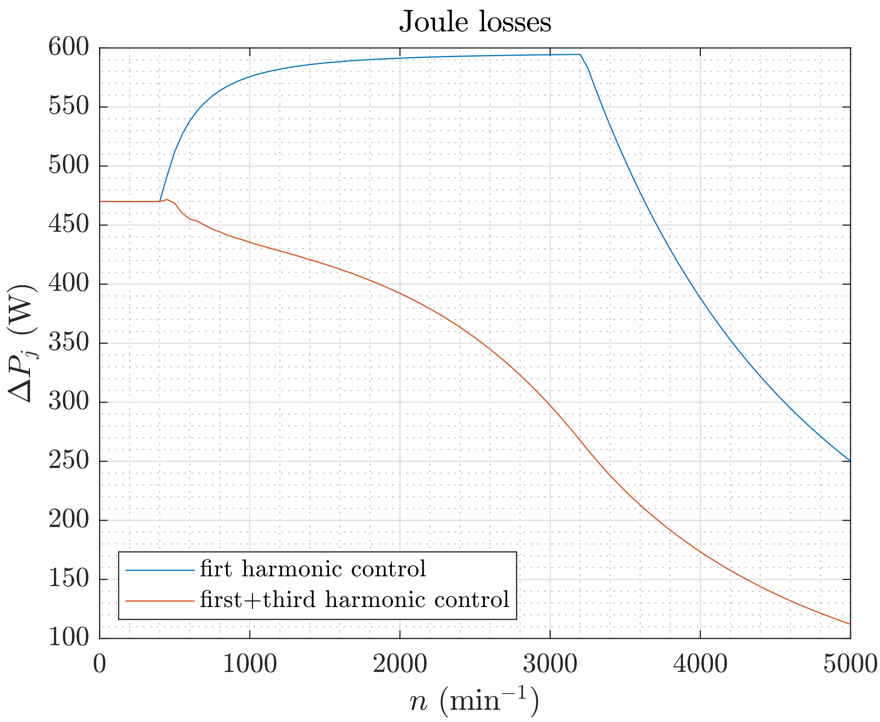

Fig. 4.7: Comparison of Joule losses of only first harmonic and first+third harmonic control for the same torque-speed characteristics (equal to maximum torque of the first harmonic control)

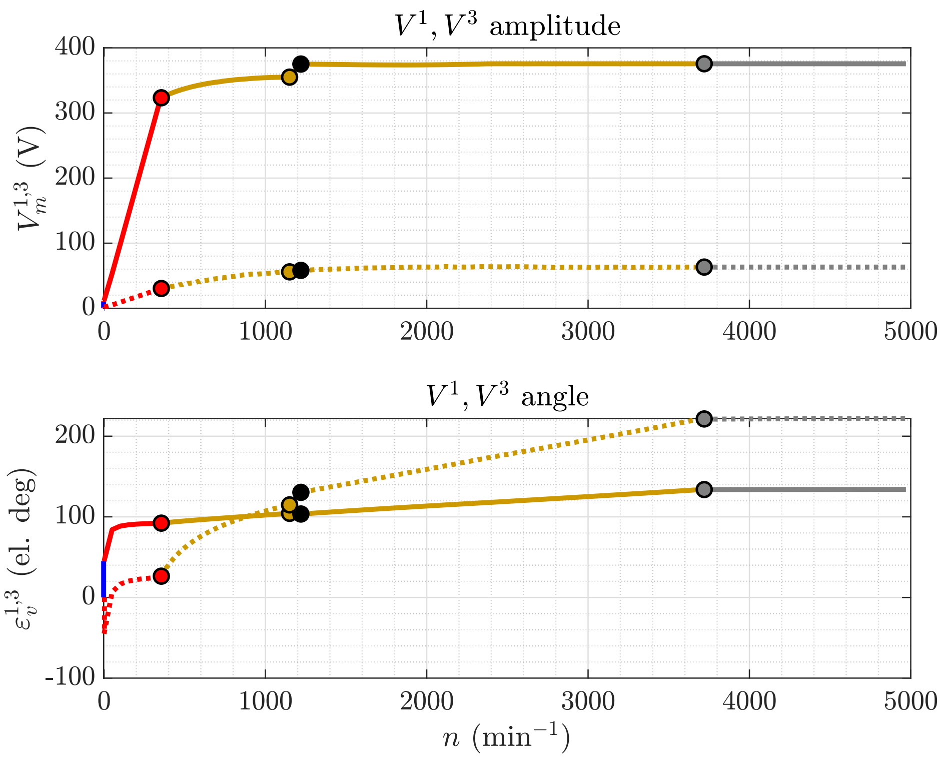

Fig. 4.6: Voltage amplitude and angle of first harmonic (solid line) and third harmonic (dashed line) component

5 Conclusion

This report brings a comprehensive summary of the operations of a generally m-phase induction machine in all operation strategies relating to the stator current optimization.

The second part of this report is focused on optimal control with the combination of the first and third harmonic control of the multiphase ($m > 3$) induction machine. The boundary conditions define only the limit of maximal amplitude of both current and voltage. It is proven that **until the current or voltage limit is not met, it is not beneficial to apply the third harmonic currents** (expecting realistic equivalent circuit parameters).

The maximum torque-speed characteristic requires both first and third harmonic currents for all speeds and control strategies (MTPA II, MC&MT, and MTPV). There is obviously a large **increase** in the **maximum power-speed characteristic**, more than 30 % for every speed point. It is caused by the increased maximum value of rms of current or voltage or both.

If the same torque-speed characteristics for both control types are compared, the decrease of the Joule losses (proportional to the rms value of current) is significant.

Since the combination of the first+third harmonic control of multiphase IM seems very promising, we can define three points that can partially devaluate its benefits (excluding the problems with the power converter, switching frequency, number of conductors, etc.):

1. **Increase of iron losses**. The complicated current and voltage waveforms induce higher harmonic components in the magnetic flux and flux density which can generally increase the iron losses, which can partially compensate for the effect of decreased Joule losses.

2. **Nonlinear parameters**. All of the calculations were expecting the constant equivalent circuit parameters. The magnetizing inductance is usually mostly affected by the saturation of the magnetic circuit which leads to the deviation from the MTPA to obtain the most optimal torque. We cannot guess now how the parameters will change during the first+third harmonic current control.

3. **Different maximum voltage**. If we start to think about the multiphase machine as a counterpart to the existing (or designed) solution with a three-phase induction machine, the discussion about the maximum available voltage is inevitable. Since the input voltage of the DC link is expected to act similarly for all machines (simplified as a constant), the amplitude of the phase voltage is increased by the non-rotating component (m-th harmonic) by the space vector PWM or triangular zero-sequence PWM techniques. Practically, without these techniques, the amplitude of the phase voltage equals $V_{DC}/2$. Application of these techniques increases the motor voltage and defines the amplitude of the line-line voltage equal to V_{DC}. An increase in the phase voltage is then 15.5 %, 5.1 %, 2.6 %, and 1.5 % for 3, 5, 7, and 9-phase machines respectively. But the multiphase machines are generally designed with the higher winding factor and the decrease of the phase voltage is compensated.

References

[1] LAKSAR, Jan; ŠMÍDL, Václav; KOMRSKA, Tomáš; ADAM, Lukáš. *Optimal Current Setpoints for Five-phase Synchronous Drive.* IECON 2023 The 49th Annual Conference of the IEEE Industrial Electronics Society.

[2] ELDEEB, Hisham; HACKL, Christoph M; HORLBECK, Lorenz; KULLICK, Julian. *A unified theory for optimal feedforward torque control of anisotropic synchronous machines.* International Journal of Control. 2018, vol. 91, no. 10, pp. 2273–2302.

[3] KALAJ, Patrik; KOMRSKA, Tomáš; KINDL, Vladimír, et al. *Multi-Pole Winding Behavior in Multiphase Motors Under Current Harmonics Operation.* IEEE Transactions on Energy Conversion. 2022, vol. 37, no. 4, pp. 2546–2555. Address: ⟨`http://dx.doi.org/10.1109/TEC.2022.3183262`⟩.

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Revision history

Rev.	Chapter	Description of change	Date	Name
1	All	Document release	1.11.2023	Jan Laksar